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A family of continued fractions

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ABSTRACT

We investigate a one-parameter family of infinite generalised continued fractions. The fractions converge to rational values which can be explicitly evaluated. The sequences of numerators and denominators are already to be found in the Online Encyclopedia of Integer Sequences with reference to various topics – but not to continued fractions.

1. Introduction

A continued fraction is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}},$$

written more compactly as

$$a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\,\cdots.$$

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We shall require the a_k and b_k to be integers, all strictly positive except possibly for a_0. In fact, the continued fractions under consideration in the present paper will always have $a_0 = 0$. If each b_k is 1 the expression is referred to as a *simple* continued fraction. Truncating a continued fraction will clearly produce a rational number, its kth *convergent*

$$\frac{p_k}{q_k} = a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\cdots\frac{b_k}{a_k},$$

and the value of the infinite continued fraction is by definition $\lim_{k\to\infty} p_k/q_k$, provided the limit exists. It is well known that an infinite simple continued fraction always converges to an irrational limit. In the general case, however, the limit need not exist: it is possible that the convergents oscillate between two accumulation points. An example of a continued fraction for which the two accumulation points can be determined is given in [2]. Moreover, even if the limit does exist, it need not be irrational. However there are various sufficient conditions for the existence of the limit; and the very simple condition that $a_k \geqslant b_k$ for all sufficiently large k, according to Chrystal [3] originally due to Legendre, guarantees that the limit exists and is irrational. See, for example, [1, pages 149–150] or [3, pages 506–509].

Let s be a positive integer and consider the continued fraction

$$\frac{1+s}{1+}\,\frac{2+s}{2+}\,\frac{3+s}{3+}\,\cdots,$$

which, as we shall show, converges. This fraction clearly does not satisfy the above condition and so there is no particular reason to suppose that it is irrational; all the same, "that's the way you bet". So it comes as something of a surprise to find that the continued fraction is in fact rational for all s, and that we can evaluate it quite simply. Moreover, the sequences of numerators and denominators, while not "top ten" familiar sequences, do occur in the literature, and provide the answers to certain questions concerning combinatorics and formal languages. Curiously, it is known that the continued fraction in question is not rational when $s = 0$: specifically,

$$\frac{1}{1+}\,\frac{2}{2+}\,\frac{3}{3+}\,\cdots = \frac{1}{e-1},$$

a result proved by Euler (*Opera Omnia*, series 1, volume 8, chapter XVIII; English translation [4, page 314]).

Our main result is the following.

Theorem 1. *For any positive integer s the continued fraction*

$$\frac{1+s}{1+}\,\frac{2+s}{2+}\,\frac{3+s}{3+}\,\cdots$$

converges. Denote its limit by $f(s)$, and define a sequence of polynomials recursively by

$$w(z;1) = 1, \qquad w(z;2) = z+4,$$
$$w(z;s) = (z+2s)w(z;s-1) - (s-2)(z+s)w(z;s-2) \quad \textit{for } s \geqslant 3.$$

Then

$$f(s) = \frac{w(0;s)}{w(-1;s)}$$

is a rational number.

The key step in proving Theorem 1 is to derive an identity between the values of $w(z; s)$ for fixed s and varying z.

Lemma 1. *For $s \geqslant 2$ we have the polynomial identities*

$$w(z; s) - w(z-1; s) - (s-1)w(z; s-1) = 0, \tag{1}$$

$$w(z; s-1) - w(z-1; s) + (z+s)w(z-1; s-1) = 0. \tag{2}$$

Proof. The identities are easily confirmed when $s = 2$; assume that they are true for some specific integer $s - 1 \geqslant 2$. That is,

$$w(z; s-1) - w(z-1; s-1) - (s-2)w(z; s-2) = 0, \tag{3}$$

$$w(z; s-2) - w(z-1; s-1) + (z+s-1)w(z-1; s-2) = 0; \tag{4}$$

we also have by definition

$$w(z; s) - (z+2s)w(z; s-1) + (s-2)(z+s)w(z; s-2) = 0, \tag{5}$$

and since this is true identically in z we may replace z by $z - 1$ to obtain

$$w(z-1; s) - (z-1+2s)w(z-1; s-1) + (s-2)(z-1+s)w(z-1; s-2) = 0. \tag{6}$$

Denote the left-hand sides of Eqs. (1)–(6) by $L_1, \ldots, L_6$. Then

$$L_1 = (z+1+s)L_3 + (s-2)L_4 + L_5 - L_6 = 0,$$

$$L_2 = L_3 + (s-2)L_4 - L_6 = 0,$$

which completes the proof of the two recurrences. $\square$

The identity which is actually important is the next one.

Lemma 2. *For any $s \geqslant 1$ we have the equality of polynomials*

$$w(z; s) = -zw(z-1; s) + (z+s)w(z-2; s).$$

Proof. The result is true for $s = 1$ and for $s = 2$. For any particular integer $s \geqslant 3$, suppose that the identity is true for $s - 2$ and $s - 1$; we need to show that

$$w(z; s) + zw(z-1; s) - (z+s)w(z-2; s) = 0.$$

Using the definition three times and rearranging, we have

$$\begin{aligned}
\mathrm{LHS} &= \big((z+2s)w(z; s-1) - (s-2)(z+s)w(z; s-2)\big) \\
&\quad + z\big((z-1+2s)w(z-1; s-1) - (s-2)(z-1+s)w(z-1; s-2)\big) \\
&\quad - (z+s)\big((z-2+2s)w(z-2; s-1) - (s-2)(z-2+s)w(z-2; s-2)\big) \\
&= (z+2s)\big[w(z; s-1) + zw(z-1; s-1) - (z+s-1)w(z-2; s-1)\big]
\end{aligned}$$

$$-(s-2)(z+s)\big[w(z;s-2)+zw(z-1;s-2)-(z+s-2)w(z-2;s-2)\big]$$
$$-z\big[w(z-1;s-1)-w(z-2;s-1)-(s-2)w(z-1;s-2)\big].$$

Here the first term in square brackets is zero by the inductive assumption for $s-1$; the second is zero by the inductive assumption for $s-2$; and the third is zero by the first result of Lemma 1. This completes the proof. □

Proof of Theorem 1. As is well known (see, for example, [1]), the convergents p_k/q_k to a generalised continued fraction

$$a_0+\frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\cdots$$

can be defined by the recurrences $p_k=a_kp_{k-1}+b_kp_{k-2}$ and $q_k=a_kq_{k-1}+b_kq_{k-2}$, with initial conditions $p_{-2}=q_{-1}=0$ and $p_{-1}=q_{-2}=1$. (The recurrences do not necessarily, however, give the fractions p_k/q_k in lowest terms.) In the present instance we have $a_0=0$, $a_k=k$ and $b_k=k+s$; since we are not interested in the case $k=-2$ the relations may be written

$$p_{-1}=q_0=1,\qquad p_0=q_{-1}=0,$$
$$p_k=kp_{k-1}+(k+s)p_{k-2},\qquad q_k=kq_{k-1}+(k+s)q_{k-2}.$$

We note that the conditions on q_k imply $q_k\geqslant k!$ for all $k\geqslant 0$. For convenience we write

$$u(s)=w(0;s)\quad\text{and}\quad v(s)=w(-1;s).$$

Then for every $k\geqslant -1$ and every $s\geqslant 1$ the identity

$$u(s)q_k-v(s)p_k=(-1)^k w(k;s)$$

holds; the proof, by induction on k, is a straightforward application of Lemma 2. Thus we have

$$\frac{u(s)}{v(s)}-\frac{p_k}{q_k}=(-1)^k\frac{w(k;s)}{v(s)q_k};$$

for any fixed s the right-hand side vanishes as $k\to\infty$, because the numerator is a polynomial in k and the denominator is at least $k!$. Hence

$$f(s)=\lim_{k\to\infty}\frac{p_k}{q_k}=\frac{u(s)}{v(s)}=\frac{w(0;s)}{w(-1;s)}.$$

Finally, it is clear from the definition that every $w(z;s)$ is a polynomial with integral coefficients, so the numerator and denominator of this last fraction are integers and the proof is complete. □

2. The sequences $u(s)$ and $v(s)$

The recurrence for $w(z;s)$ carries over immediately to $u(s)$ and $v(s)$, so that we have

$$u(1)=1, \qquad u(2)=4, \qquad u(s)=2su(s-1)-s(s-2)u(s-2),$$
$$v(1)=1, \qquad v(2)=3, \qquad v(s)=(2s-1)v(s-1)-(s-1)(s-2)v(s-2).$$

Thus the continued fraction $f(s)$ has the values

$$\frac{1}{1}, \quad \frac{4}{3}, \quad \frac{21}{13}, \quad \frac{136}{73}, \quad \frac{1045}{501}, \quad \frac{9276}{4051}, \quad \frac{93289}{37633}$$

for $s=1,2,3,4,5,6,7$. We shall prove that for each s the fraction $u(s)/v(s)$ is in lowest terms.

The sequences $u(s)$ and $v(s)$ have been studied before, and are in fact Sloane's A052852 and A000262, as is easily verified by comparing the above recurrences and initial conditions with those given in the OEIS [5]. In this section we survey some properties of these sequences, which are of interest in that they have no apparent connection with continued fractions, and thereby furnish an unexpected link between disparate areas of mathematics. We begin by exhibiting a coupled pair of first-order recurrences for $u(s)$ and $v(s)$, namely,

$$u(s)=su(s-1)+sv(s-1) \quad \text{and} \quad v(s)=u(s-1)+sv(s-1), \tag{7}$$

with initial conditions $u(1)=v(1)=1$. These can be neatly expressed in matrix form and then iterated to give

$$\begin{aligned}\begin{pmatrix}u(s)\\ v(s)\end{pmatrix} &= \begin{pmatrix}s & s\\ 1 & s\end{pmatrix}\begin{pmatrix}u(s-1)\\ v(s-1)\end{pmatrix}\\ &= \begin{pmatrix}s & s\\ 1 & s\end{pmatrix}\begin{pmatrix}s-1 & s-1\\ 1 & s-1\end{pmatrix}\cdots\begin{pmatrix}3 & 3\\ 1 & 3\end{pmatrix}\begin{pmatrix}2 & 2\\ 1 & 2\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}.\end{aligned}$$

We can also interpret $u(s)$ and $v(s)$ as the answers to two related combinatorial problems. Let s be a positive integer and suppose that $s-1$ people attend a party. (Rather a poor party if $s=1$; probably an even worse one if $s=2$.) There are $s+1$ seats available, but it is not necessary for everyone to be seated. Let $U(s)$ be the number of seating arrangements possible; if the number of available seats is reduced to s let the number be $V(s)$. Then $U(s)=u(s)$ and $V(s)=v(s)$. To see this, suppose that we wish to seat at most $s-1$ people on $s+1$ seats. We can either

- leave the last seat empty: then $s-1$ people can occupy the remaining s seats in $V(s)$ ways; or
- choose one of the $s-1$ people for the last seat: then allocate $s-2$ people to s seats in $U(s-1)$ ways.

Hence

$$U(s)=V(s)+(s-1)U(s-1).$$

If we wish to place at most $s-1$ people on s seats we can

- leave the last person without a seat and place at most $s-2$ people onto s seats: this can be done in $U(s-1)$ ways; or
- choose one of the s seats for the last person, then allocate at most $s-2$ people to $s-1$ seats in $V(s-1)$ ways.

Therefore

$$V(s) = U(s-1) + sV(s-1).$$

It is easy to see that these two relations imply

$$U(s) = sU(s-1) + sV(s-1),$$

and that $U(1) = V(1) = 1$; hence U and V satisfy the same initial values and recurrences as u and v, and this justifies our claim. A more formal statement of the result is that $u(s)$ is the number of one-to-one partial functions from $\{1, \dots, s-1\}$ to $\{1, \dots, s+1\}$ and $v(s)$ the number from $\{1, \dots, s-1\}$ to $\{1, \dots, s\}$.

A generalisation of these problems is considered under sequence A086885 in [5], where $T(i, j)$ is defined to be the number of ways in which at most j people can occupy $i \geqslant j$ seats. The recurrence

$$T(i, j) = T(i, j-1) + iT(i-1, j-1) \tag{8}$$

is given, which is easily proved by the above arguments; if $i = s$ and $j = s-1$ we have precisely our recurrence for $v(s)$. In fact the general seating problem can be solved in terms of our polynomials $w(z; s)$, since

$$T(i, j) = w(i-j-2; j+1):$$

to prove this, substitute $s = j+1$ and $z = i-j-1$ into (2) to obtain a recurrence of the form (8). As a result of the combinatorial interpretations we can write down the explicit formulae

$$u(s) = \sum_{j=0}^{s-1} j! \binom{s-1}{j} \binom{s+1}{j} \quad \text{and} \quad v(s) = \sum_{j=0}^{s-1} j! \binom{s-1}{j} \binom{s}{j}. \tag{9}$$

Now it is easy to see that binomial coefficients have the divisibility properties

$$r \,\Big|\, j! \binom{r}{j} \quad \text{for } j \geqslant 1 \quad \text{and} \quad r-1 \,\Big|\, j! \binom{r}{j} \quad \text{for } j \geqslant 2.$$

Therefore the identities (9) give the congruences

$$u(s) \equiv 1 \pmod{s-1} \quad \text{and} \quad u(s) \equiv 1 + (s-1)(s+1) \equiv 0 \pmod{s},$$

the latter being in any case obvious from the first of the recurrences (7), as well as

$$v(s) \equiv 1 \pmod{s-1} \quad \text{and} \quad v(s) \equiv 1 \pmod{s}.$$

Using these congruences we can prove the result foreshadowed on page 908.

Lemma 3. *For any $s \geqslant 1$ the integers $u(s)$ and $v(s)$ are relatively prime.*

Proof. Suppose that p is a common prime factor of $u(s)$ and $v(s)$. Then p is a factor of $u(s) - v(s)$ and of $sv(s) - u(s)$; from the recurrences (7) we have

$$p \mid (s-1)u(s-1) \quad \text{and} \quad p \mid s(s-1)v(s-1).$$

But from the congruences just noted, p is not a factor of s, nor of $s-1$, and so p is a common factor of $u(s-1)$ and $v(s-1)$. Proceeding inductively we see that p must be a factor of $u(1)$ and $v(1)$, which is impossible. Hence there is no such p and the result is proved. □

To conclude this section we mention that the OEIS [5] cites $u(s)$ as the number of words in a certain language on s symbols and gives a determinant formula and numerous combinatorial interpretations for $v(s)$. There are also connections with Lah numbers and Laguerre polynomials.

3. The asymptotic behaviour of $f(s)$

The OEIS [5] gives a somewhat inscrutable asymptotic formula for $v(s)$, and none for $u(s)$. However it is not hard to determine the asymptotic behaviour of the quotient $f(s) = u(s)/v(s)$. From (7) we obtain for $f(s)$ the recurrence relation

$$f(s) = \frac{sf(s-1) + s}{f(s-1) + s} \tag{10}$$

for $s \geqslant 2$, with the initial condition $f(1) = 1$, and this can be used to find upper and lower bounds for $f(s)$.

Lemma 4. *For any $s \geqslant 1$ we have $\sqrt{s} - 1 < f(s) \leqslant \sqrt{s}$.*

Proof. The result is clear when $s = 1$. Let $s \geqslant 2$ and assume that the inequalities are true for $s-1$. Then

$$\sqrt{s-1} - 1 < f(s-1) \leqslant \sqrt{s-1} < \sqrt{s};$$

by using the recurrence (10) we have

$$f(s) = s\left(1 - \frac{s-1}{f(s-1) + s}\right) < s\left(1 - \frac{s-1}{s + \sqrt{s}}\right) = \sqrt{s}$$

and

$$f(s) > \frac{s\sqrt{s-1}}{s + \sqrt{s}} = \frac{\sqrt{s}\sqrt{s-1}}{\sqrt{s} + 1} > \frac{s-1}{\sqrt{s} + 1} = \sqrt{s} - 1.$$

The proof is complete. □

Corollary. *$f(s) \sim \sqrt{s}$ as $s \to \infty$.*

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